

On tight spans and tropical polytopes for directed distances

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April 2010

Abstract

An extension (V, d) of a metric space (S, μ) is a metric space with $S \subseteq V$ and $d|_S = \mu$, and is said to be *tight* if there is no other extension (V, d') of (S, μ) with $d' \leq d$. Isbell and Dress independently found that every tight extension is isometrically embedded into a certain metrized polyhedral complex associated with (S, μ) , called the *tight span*. This paper develops an analogous theory for directed metrics, which are “not necessarily symmetric” distance functions satisfying the triangle inequality. We introduce a directed version of the tight span and show that it has such a universal embedding property for tight extensions. Also we newly introduce another natural class of extensions, called *cyclically tight extensions*, and show that (a fiber of) the tropical polytope, introduced by Develin and Sturmfels, has a universal embedding property for cyclically tight extensions. As an application, we prove the following directed version of tree metric theorem: directed metric μ is a directed tree metric if and only if the tropical rank of $-\mu$ is at most two. Also we describe how tight spans and tropical polytopes are applied to the study in multicommodity flows in directed networks.

1 Introduction

An *extension* (V, d) of a metric space (S, μ) is a metric space with $S \subseteq V$ and $d|_S = \mu$, and is said to be *tight* if there is no other extension (V, d') of (S, μ) with $d' \leq d$. Isbell [10] and Dress [6] independently found that there exists a *universal* tight extension with property that it contains every tight extension as a subspace. The universal tight extension is realized by the set of minimal elements of the polyhedron

$$\{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s, t) \ (s, t \in S)\}$$

in (\mathbf{R}^S, l_∞) , which is called the *injective hull* or the *tight span*. The tight span reflects, in geometric way, several hidden combinatorial properties of μ . For example, μ is a tree metric if and only if its tight span is a tree (one-dimensional contractible complex); see [6, Theorem 8]. This property reinterprets the classical *tree metric theorem* [2, 16, 18], and has a significance in the phylogenetic tree reconstruction in the biological sciences. Tight spans also arise in quite different contexts, including the *k-server problem* in computer sciences [3] and the *multicommodity flows problems* in combinatorial optimization [8, 11].

In applications to both areas, the geometry of tight spans and the above-mentioned universality play crucial roles.

The goal of this paper is to develop an analogous theory of tight spans for directed distances and metrics. A *directed distance* d on a set V is a nonnegative real-valued function $d : V \times V \rightarrow \mathbf{R}_+$ having zero diagonals $d(x, x) = 0$ ($x \in V$). A directed distance d is said to be a *directed metric* if it satisfies the triangle inequalities $d(x, y) + d(y, z) \geq d(x, z)$ ($x, y, z \in V$). In both definitions, we do not impose the symmetric condition $d(x, y) = d(y, x)$. A pair (V, d) of a set V and its directed metric d is called a *directed metric space*. We can define the concept of tight extensions in the same way. The motivation for our theory comes from the following natural questions:

- *Does there exist the universal tight extension?*
- *If exists, does its geometry characterize a class of combinatorial distances, related to directed versions of tree metrics?*
- *Are there reasonable applications to the study of multicommodity flows in directed networks?*

The main contribution of this paper answers these questions affirmatively. In particular a directed analogue of the tight span does exist. Furthermore we newly introduce another natural class of extensions, called *cyclically tight extensions*. Interestingly, it has an unexpected connection to the theory of *tropical polytopes*, developed by Develin and Sturmfels [5]. In fact, we show that the tropical polytope generated by the distance matrix of μ has the universal embedding property for cyclically tight extensions.

In Section 2, for a directed distance μ , we define a directed analogue T_μ of the tight span (the *directed tight span*) as the set of minimal points in an unbounded polyhedron P_μ associated with μ . The tropical polytope is the projection \bar{Q}_μ of the set Q_μ of minimal points in another polyhedron associated with μ . We introduce a well-behaved *section*, a subset in Q_μ projected bijectively into \bar{Q}_μ , called a *balanced section*. We endow T_μ , Q_μ , and any balanced section R with a directed analogue of the l_∞ -metric. We investigate the behaviors of these directed metric spaces, which is placed in the main body of Section 2. Then we prove the universal embedding properties that every tight extension of a metric μ is isometrically embedded into the tight span T_μ , and every cyclically tight extension is isometrically embedded into a balanced section of Q_μ (Theorem 2.12).

In Section 3, as an application of the embedding property and the dimension criteria, we prove that some classes of distances realized by *oriented trees* can be characterized by one-dimensionality of tight spans and tropical polytopes (Theorems 3.1 and 3.2). In particular, we prove the following directed version of the tree metric theorem: *a directed metric μ is a directed tree metric if and only if the tropical rank of $-\mu$ is at most 2* (Corollary 3.3). This complements the argument in [5, Section 5].

In Section 4, we briefly sketch how tight spans and tropical polytopes are applied to the study of directed multicommodity flow problems. We prove that the linear programming dual to multicommodity flow problems reduces to *facility location problems* on tight spans and tropical polytopes (Theorems 4.1 and 4.2). Further study of this approach will be given in the next paper [9].

Notation. The sets of real numbers and nonnegative real numbers are denoted by \mathbf{R} and \mathbf{R}_+ , respectively. For a set X , the sets of functions from X to \mathbf{R} and from X to \mathbf{R}_+ are denoted by \mathbf{R}^X and \mathbf{R}_+^X , respectively. For a subset $Y \subseteq X$, the characteristic function $\mathbf{1}_Y \in \mathbf{R}^X$ is defined by $\mathbf{1}_Y(x) = 1$ for $x \in Y$ and $\mathbf{1}_Y(x) = 0$ for $x \notin Y$. We particularly denote by $\mathbf{1}$ the all-one function $\mathbf{1}_X$ in \mathbf{R}^X . The singleton set $\{s\}$ is often

denoted by s , such as $\mathbf{1}_s$ instead of $\mathbf{1}_{\{s\}}$. For $p, q \in \mathbf{R}^X$, $p \leq q$ means $p(x) \leq q(x)$ for each $x \in X$, and $p < q$ means $p(x) < q(x)$ for each $x \in X$. For $p \in \mathbf{R}^X$, $(p)_+$ is a point in \mathbf{R}^X defined by $((p)_+)(x) = \max\{p(x), 0\}$ for $x \in X$. For a set P in \mathbf{R}^X , a point p in P is said to be *minimal* if there is no other point $q \in P \setminus p$ with $q \leq p$.

In this paper, a directed metric (distance) is often simply called a metric (distance). A metric (distance) in the ordinary sense is particularly called an *undirected metric* (*undirected distance*). We use the terminology of undirected metrics in an analogous way (we do not know any reference including a systematic treatment for directed metrics). For two directed metric spaces (V, d) and (V', d') , an *isometric embedding* from (V, d) to (V', d') is a map $\rho : V \rightarrow V'$ satisfying $d'(\rho(x), \rho(y)) = d(x, y)$ for all (ordered) pairs $x, y \in V$. For a directed metric d on V , and two subsets $A, B \subseteq V$, let $d(A, B)$ denote the minimum distance from A to B :

$$d(A, B) = \inf\{d(x, y) \mid (x, y) \in A \times B\}.$$

In our theory, the following directed metric D_∞^+ on \mathbf{R}^X is particularly important:

$$D_\infty^+(p, q) = \|(q - p)_+\|_\infty \quad (= \max_{x \in X} (q - p)_+(x)) \quad (p, q \in \mathbf{R}^X).$$

For an directed or undirected graph G , its vertex set and edge set are denoted by VG and EG , respectively. If directed, an edge with tail x and head y is denoted by xy . If undirected, we do not distinguish xy and yx .

2 Tight spans and tropical polytopes

In this section, we introduce and study the tight span and the tropical polytope of a directed distance. Let μ be a directed distance on a finite set S . Let S^c and S^r be copies of S . For an element $s \in S$, the corresponding elements in S^c and S^r are denoted by s^c and s^r , respectively. We denote $S^c \cup S^r$ by S^{cr} . For a point $p \in \mathbf{R}^{S^{cr}}$, the restrictions of p to S^c and S^r are denoted by p^c and p^r , respectively, i.e., $p = (p^c, p^r)$. We define the following polyhedral sets:

$$\begin{aligned} \Pi_\mu &= \{p \in \mathbf{R}^{S^{cr}} \mid p(s^c) + p(t^r) \geq \mu(s, t) \ (s, t \in S)\}. \\ P_\mu &= \Pi_\mu \cap \mathbf{R}_+^{S^{cr}}. \\ T_\mu &= \text{the set of minimal elements of } P_\mu. \\ Q_\mu &= \text{the set of minimal elements of } \Pi_\mu. \\ Q_\mu^+ &= Q_\mu \cap \mathbf{R}_+^{S^{cr}}. \end{aligned}$$

We call T_μ the *directed tight span*, or simply, the *tight span*. The polyhedron Π_μ has the linearity space $(\mathbf{1}, -\mathbf{1})\mathbf{R}$. The natural projection of vector $v \in \mathbf{R}^{S^{cr}}$ to $\mathbf{R}^{S^{cr}}/(\mathbf{1}, -\mathbf{1})\mathbf{R}$ is denoted by \bar{v} . The projection \bar{Q}_μ of Q_μ coincides with the *tropical polytope* generated by S by S matrix $(-\mu(s, t) \mid s, t \in S)$; see Develin and Sturmfels [5]. We note the following relation:

$$\begin{array}{ccccc} Q_\mu & \supseteq & Q_\mu^+ & \subseteq & T_\mu \subseteq P_\mu \\ & \downarrow & \swarrow & & \\ & \bar{Q}_\mu & & & \end{array}$$

Here the arrow means the projection. In the inclusions, Q_μ^+ is a subcomplex of T_μ , and T_μ is a subcomplex of P_μ .

A subset $R \subseteq Q_\mu$ is called a *section* if the projection $p \in R \mapsto \bar{p}$ is bijective. A subset $R \subseteq \mathbf{R}^{S^{cr}}$ is said to be *balanced* if there is no pair p, q of points in R such that $p^c < q^c$

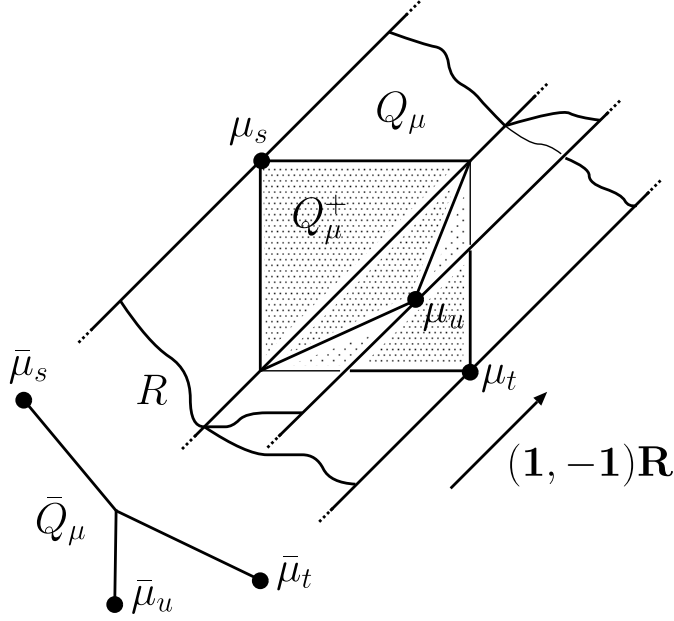


Figure 1: Q_μ , Q_μ^+ , and \bar{Q}_μ

or $p^r < q^r$. We are mainly interested in balanced sections. We metrize these subsets as follows. Let D_∞ be a directed metric on \mathbf{R}^{scr} defined as

$$D_\infty(p, q) = \max\{D_\infty^+(p^c, q^c), D_\infty^+(q^r, p^r)\} \quad (p, q \in \mathbf{R}^{scr}).$$

As subspaces of $(\mathbf{R}^{scr}, D_\infty)$, we obtain directed metric spaces (T_μ, D_∞) , (Q_μ, D_∞) , (Q_μ^+, D_∞) , and (R, D_∞) for any balanced section $R \subseteq Q_\mu$. Figure 1 illustrates those subsets for all-one distance μ on a 3-set $S = \{s, t, u\}$. In this case, $T_\mu = Q_\mu^+$ (accidentally), Q_μ is the union of three infinite strips with a common side, Q_μ^+ is a *folder* consisting of three triangles, and \bar{Q}_μ is a star of three leaves. Here μ_s is a vector in \mathbf{R}^{scr} consisting of the s -th column and the s -th row of μ ; see Section 2.3 for definition.

The rest of this section is organized as follows. In the next Section 2.1, we present basic properties of these polyhedral sets. In Section 2.2, we prove the existence of certain *nonexpansive retractions* among them. This is the most substantial part of this section. In Section 2.3, we show that T_μ, Q_μ, Q_μ^+ , and any balanced section are *geodesic* in the *directed* sense. Also we show that distance μ is *embedded into* them. In Section 2.4, we introduce notions of tight extensions and cyclically tight extensions of a directed metric space (S, μ) , and prove that T_μ and Q_μ^+ have the universal embedding properties for tight extensions and cyclically tight extensions, respectively. In Section 2.5, we present dimension criteria for T_μ and \bar{Q}_μ . In the sequel, μ is supposed to be a distance on a finite set S , without noted.

2.1 Basic properties

Here we briefly summarize some fundamental properties of T_μ , Q_μ , and Q_μ^+ . For a point $p \in \mathbf{R}^{scr}$, we denote by $K(p)$ the bipartite graph with bipartition (S^c, S^r) and edge set $EK(p) = \{s^c t^r \mid s^c \in S^c, t^r \in S^r, p(s^c) + p(t^r) = \mu(s, t)\}$. The minimality of p can be rephrased in terms of $K(p)$ as follows.

Lemma 2.1. (1) *A point p in P_μ belongs to T_μ if and only if $K(p)$ has no isolated vertices u with $p(u) > 0$.*

(2) A point p in Π_μ belongs to Q_μ if and only if $K(p)$ has no isolated vertices.

The next lemma says that the projections of T_μ and of Q_μ to \mathbf{R}^{S^c} (or \mathbf{R}^{S^r}) are isometries. So we can consider T_μ , Q_μ , and Q_μ^+ in $(\mathbf{R}^{S^c}, D_\infty^+)$.

Lemma 2.2. (1) $D_\infty(p, q) = D_\infty^+(p^c, q^c) = D_\infty^+(q^r, p^r)$ for $p, q \in T_\mu$ and for $p, q \in Q_\mu$.

(2) For $p, q \in Q_\mu$, $p^c < q^c$ if and only if $p^r > q^r$.

Proof. (1). Suppose $D_\infty^+(p^c, q^c) = q(s^c) - p(s^c) > 0$ for $s^c \in S^c$. By Lemma 2.1, there is $t^r \in S^r$ such that $q(s^c) + q(t^r) = \mu(s, t)$. Hence, $q(s^c) - p(s^c) = \mu(s, t) - q(t^r) - p(s^c) \leq p(t^r) - q(t^r) \leq D_\infty^+(q^r, p^r)$. Thus $D_\infty^+(p^c, q^c) \leq D_\infty^+(q^r, p^r)$. Similarly $D_\infty^+(p^c, q^c) \geq D_\infty^+(q^r, p^r)$.

(2). Suppose $p^c < q^c$. If $p(s^r) \leq q(s^r)$ for some $s^r \in S^r$, then there is $t^c \in S^c$ with $q(t^c) + q(s^r) = \mu(t, s)$, and $p(t^c) + p(s^r) < q(t^c) + q(s^r) = \mu(t, s)$; a contradiction to $p \in Q_\mu$. \square

Next we study the local structure of T_μ . For a point $p \in T_\mu$, let $F(p)$ denote the face of T_μ that contains p in its relative interior. Suppose that $K(p)$ has k (bipartite) components having no vertices $u \in S^{cr}$ with $p(u) = 0$. Let $(A_1^c, B_1^r), \dots, (A_k^c, B_k^r)$ be bipartitions of these components. Then the affine span of $F(p)$ is given by

$$(2.1) \quad p + \sum_{i=1}^k x_i (\mathbf{1}_{A_i^c}, -\mathbf{1}_{B_i^r}) \quad (x_1, \dots, x_k \in \mathbf{R}).$$

Therefore the dimension of $F(p)$ is given as follows; compare [5, Proposition 17].

Lemma 2.3. For a point $p \in T_\mu$, the dimension of $F(p)$ is equal to the number of connected components of $K(p)$ having no vertex s with $p(s) = 0$.

We end this subsection with a seemingly obvious lemma.

Lemma 2.4. For any balanced set $B \subseteq Q_\mu$, there exists a balanced section $R \subseteq Q_\mu$ containing B . If $B \subseteq Q_\mu^+$, then there exists a balanced section $R \subseteq Q_\mu^+$ containing B .

Proof. Let B be a balanced set in Q_μ . By Zorn's lemma, we can take an inclusion-maximal balanced set $R \subseteq Q_\mu$ containing B . We show that R is a section. Suppose not. Then there is $p_0 \in Q_\mu$ with $\{p_0 + (\mathbf{1}, -\mathbf{1})\mathbf{R}\} \cap B = \emptyset$. Let $L = p_0 + (\mathbf{1}, -\mathbf{1})\mathbf{R}$. For each point $q \in \mathbf{R}^{S^{cr}}$, let L_q be the set of points p in L such that $\{p, q\}$ is balanced. Then L_q is a closed segment in L ; see below. We claim:

(*) For every balanced pair $q, q' \in \mathbf{R}^{S^{cr}}$, the intersection $L_q \cap L_{q'}$ is nonempty.

Suppose true. By Helly property of segments on a line, the intersection $\bigcap \{L_q \mid q \in B\}$ is nonempty. Take any point $q \in \bigcap \{L_q \mid q \in B\}$. Then $\{q\} \cup B$ is balanced, which contradicts the maximality of B .

So we show (*). In the following, for $p \in \mathbf{R}^X$ we simply denote $\min_{x \in X} p(x)$ and $\max_{x \in X} p(x)$ by $\min p$ and $\max p$, respectively. L_q is indeed a segment given by $\{p_0 + \alpha(\mathbf{1}, -\mathbf{1}) \mid l_q \leq \alpha \leq u_q\}$, where $l_q := \min(q - p_0)^c = \min(p_0 - q)^r$ and $u_q := \max(q - p_0)^c = \max(p_0 - q)^r$. This follows from: $\{q, p_0 + \alpha(\mathbf{1}, -\mathbf{1})\}$ is balanced $\Leftrightarrow \min(p_0^c + \alpha\mathbf{1} - q^c) \leq 0$ and $\min(q^c - p_0^c - \alpha\mathbf{1}) \leq 0 \Leftrightarrow \min(p_0^r - \alpha\mathbf{1} - q^r) \leq 0$ and $\min(q^r - p_0^r + \alpha\mathbf{1}) \leq 0$ (Lemma 2.2 (2)). Therefore, if $L_q \cap L_{q'} = \emptyset$, then either $\max(q - p_0)^c < \min(q' - p_0)^c$ or $\min(q - p_0)^c > \max(q' - p_0)^c$ holds, and this implies $q^c < (q')^c$ or $q^c > (q')^c$, i.e., $\{q, q'\}$ is not balanced.

For the latter part, by the same Helly idea, it suffices to show that $L \cap Q_\mu^+$ is nonempty, and $L_q \cap Q_\mu^+$ is nonempty if $q \in Q_\mu^+$. As $L \cap Q_\mu^+ (= L \cap \mathbf{R}_+^{S^{cr}})$ is given by $\{p_0 + \alpha(\mathbf{1}, -\mathbf{1}) \mid \max -p_0^c \leq \alpha \leq \min p_0^r\}$, it is nonempty by $p_0(s^c) + p_0(t^r) \geq \mu(s, t) \geq 0$. Moreover, if $L_q \cap Q_\mu^+ = \emptyset$, then either $\min p_0^r < \min(p_0 - q)^r$ or $\max(q - p_0)^c < \max -p_0^c$ holds; both cases are impossible by $q \geq 0$. \square

2.2 Nonexpansive retractions

The goal of this subsection is to show the existence of several *nonexpansive retractions* from P_μ to T_μ , from T_μ to Q_μ^+ , and from Q_μ to any balanced section. At the first glance, they are seemingly technical lemmas. However it will turn out that many important properties can be easily derived from them. The first retraction lemma is about $P_\mu \rightarrow T_\mu$, which is a natural extension of [6, (1.9)].

Lemma 2.5. *There exists a map $\phi : P_\mu \rightarrow T_\mu$ such that*

- (1) $\phi(p) \leq p$ for $p \in P_\mu$ (and thus $\phi(p) = p$ for $p \in T_\mu$), and
- (2) $D_\infty(\phi(p), \phi(q)) \leq D_\infty(p, q)$ for $p, q \in P_\mu$.

The property (2) says that this map ϕ does not expand the distance of any pair of points in P_μ . A map between metric spaces with this property is said to be *nonexpansive*.

In a directed metric space (V, d) , there is another notion of nonexpansiveness. By a *cycle* in V we mean a cyclic permutation (p_1, p_2, \dots, p_m) of (possibly repeated) points $p_1, p_2, \dots, p_m \in V$. For a cycle $C = (p_1, p_2, \dots, p_m)$, the *length* $d(C)$ is defined by

$$d(C) = d(p_1, p_2) + d(p_2, p_3) + \dots + d(p_{m-1}, p_m) + d(p_m, p_1).$$

For a map $\varphi : V \rightarrow V$, let $\varphi(C)$ denote the cycle $(\varphi(p_1), \varphi(p_2), \dots, \varphi(p_m))$. Then there is a retraction from T_μ to Q_μ^+ not expanding the length of every cycle.

Lemma 2.6. *There exists a map $\varphi : T_\mu \rightarrow Q_\mu^+$ such that*

- (1) $\varphi(p) = p$ for $p \in Q_\mu^+$, and
- (2) $D_\infty(\varphi(C)) \leq D_\infty(C)$ for every cycle C in T_μ .

We call a map with property (2) *cyclically nonexpansive*. Any nonexpansive map is cyclically nonexpansive, and the converse is not true in general. Two notions coincide in undirected metric spaces.

Let $R \subseteq Q_\mu$ be a balanced section. For any point $p \in Q_\mu$, there uniquely exists $q \in R$ with $\bar{p} = \bar{q}$. From this correspondence, we obtain a retraction $\varphi_R : Q_\mu \rightarrow R$. Again φ_R is cyclically nonexpansive.

Lemma 2.7. *Let $R \subseteq Q_\mu$ be a balanced section.*

- (1) $D_\infty(\varphi_R(C)) \leq D_\infty(C)$ for every cycle C in Q_μ , and
- (2) subset $U \subseteq Q_\mu$ is balanced if and only if $D_\infty(\varphi_R(C)) = D_\infty(C)$ holds for every cycle C in U .

We give a remark concerning the continuity of these maps. Recall that any nonexpansive map between undirected metric spaces is always continuous (with respect to the metric topology).

Remark 2.8. One can see that map $p \mapsto \max\{(p^c)_+, (-p^r)_+\} + \max\{(-p^c)_+, (p^r)_+\}$ defines a norm $\|\cdot\|$ in $\mathbf{R}^{S^{cr}}$. The corresponding distance $\|p - q\|$ equals to $D_\infty(p, q) + D_\infty(q, p)$. Therefore any cyclically nonexpansive retraction is nonexpansive in this metric, and is continuous with respect to the topology induced by this norm. Since any norm in \mathbf{R}^n defines the same topology, it is continuous with respect to the Euclidean topology. As a consequence, all $T_\mu, Q_\mu^+, Q_\mu, \bar{Q}_\mu$ are contractible (the contractibility of tropical polytopes was shown in [5]). In particular, any balanced section is a continuous section.

The rest of this subsection is devoted to proving three lemmas. For a nonzero vector $v \in \mathbf{R}^{S^{cr}}$ and a nonnegative real $\alpha \geq 0$, let $\phi[v, \alpha] : P_\mu \rightarrow P_\mu$ be the map defined by

$$\phi[v, \alpha](p) = p + v \max\{\epsilon \in [0, \alpha] \mid p + \epsilon v \in P_\mu\} \quad (p \in P_\mu).$$

This map moves each point in P_μ toward direction v in same speed as much as possible within time α . Let $\phi[v] := \lim_{\alpha \rightarrow \infty} \phi[v, \alpha]$, which is well-defined if v has a negative component. By definition, $\phi[v]$ is identity on the set of points p such that the tangent cone of P_μ at p does not contain v . We will construct required retractions as the composition of $\phi[v]$ for several vectors v .

Proof of Lemma 2.5. For $s \in S$, let $\phi_s := \phi[-\mathbf{1}_{s^c}] \circ \phi[-\mathbf{1}_{s^r}]$. Let $S = \{s_1, s_2, \dots, s_n\}$ and let $\phi := \phi_{s_1} \circ \phi_{s_2} \circ \dots \circ \phi_{s_n}$. We remark that the map $\phi[-\mathbf{1}_u]$ decreases the u -th component of each point p as much as possible, or equivalently, until $(*)$ $p(u) = 0$ or there appears an edge covering u in $K(p)$ (Lemma 2.1 (1)). From this, we see that ϕ is a retraction from P_μ to T_μ with property (1). So it suffices to show that ϕ_s is nonexpansive for $s \in S$. Take $p, q \in P_\mu$. It suffices to compare their s^c - and s^r -components. By $(*)$ we have $\phi_s(q)(s^c) = \max\{0, \mu(s, u) - q(u^r) \mid u \in S\} = \max\{0, (\mu(s, u) - p(u^r)) + (p(u^r) - q(u^r)) \mid u \in S\} \leq \phi_s(p)(s^c) + \|(p^r - q^r)_+\|_\infty$ and $\phi_s(p)(s^r) = \max\{0, \mu(u, s) - p(u^c) \mid u \in S\} = \max\{0, (\mu(u, s) - q(u^c)) + (q(u^c) - p(u^c)) \mid u \in S\} \leq \phi_s(q)(s^r) + \|(q^c - p^c)_+\|_\infty$. Thus $(\phi_s(q)(s^c) - \phi_s(p)(s^c))_+ \leq \|(p^r - q^r)_+\|_\infty$ and $(\phi_s(p)(s^r) - \phi_s(q)(s^r))_+ \leq \|(q^c - p^c)_+\|_\infty$. Consequently we have $D_\infty(\phi_s(p), \phi_s(q)) \leq D_\infty(p, q)$. This proof method is a direct adaptation of that of [6, (1.9)].

Proof of Lemma 2.6. By Lemma 2.1, any point $p \in T_\mu \setminus Q_\mu^+$ has isolated vertices in $K(p)$. Since $\mu \geq 0$, the set I of isolated vertices in $K(p)$ belongs to either S^c or S^r . If $I \subseteq S^c$, then $p + \epsilon(\mathbf{1}_{S^c \setminus I}, -\mathbf{1}) \in T_\mu$ for small $\epsilon > 0$, and if $I \subseteq S^r$, then $p + \epsilon(-\mathbf{1}, \mathbf{1}_{S^r \setminus I}) \in T_\mu$ for small $\epsilon > 0$. On the other hand, if $p \in Q_\mu^+$, then $p + \epsilon(\mathbf{1}_{A^c}, -\mathbf{1}) \notin T_\mu$ and $p + \epsilon(-\mathbf{1}, \mathbf{1}_{A^r}) \notin T_\mu$ for any nonempty proper subset $A \subseteq S$. Define $\varphi_A^c = \phi[(\mathbf{1}_{A^c}, -\mathbf{1})]$ and $\varphi_A^r = \phi[(-\mathbf{1}, \mathbf{1}_{A^r})]$ for a nonempty proper subset $A \subseteq S$. Then both φ_A^c and φ_A^r are identity on Q_μ^+ . Order all nonempty proper subsets A_1, A_2, \dots, A_m ($m = 2^{|S|} - 2$) of S such that $A_i \subseteq A_j$ implies $i \leq j$. Consider point $p \in T_\mu$ whose $K(p)$ has nonempty set I of isolated vertices in S^c . Apply all maps $\varphi_{A_i}^c$ ($i = 1, 2, \dots, m$) to p according to the ordering of subsets A_i . In a step j , point p moves if and only if $A_j^c = S^c \setminus I$. If moves, then it moves until some $s^c \in I$ is covered by edges. So the set I of isolated vertices decreases. By the definition of the ordering, there is a step $j' > j$ such that $A_{j'}^c = S^c \setminus I$. Therefore, after the procedure, $K(p)$ has no isolated vertex, i.e., $p \in Q_\mu^+$. Motivated by this, let us define $\varphi^c = \varphi_{A_m}^c \circ \varphi_{A_{m-1}}^c \circ \dots \circ \varphi_{A_2}^c \circ \varphi_{A_1}^c$ and $\varphi^r = \varphi_{A_m}^r \circ \varphi_{A_{m-1}}^r \circ \dots \circ \varphi_{A_2}^r \circ \varphi_{A_1}^r$. By the argument above, $\varphi := \varphi^r \circ \varphi^c$ is a retraction from T_μ to Q_μ^+ . So it suffices to verify that φ_A^c is cyclically nonexpansive for each $A \subseteq S$. Take any cycle $C = (p_1, p_2, \dots, p_n)$ in T_μ . For $\alpha \geq 0$, let $\varphi_{A, \alpha}^c = \phi[(\mathbf{1}_{A^c}, -\mathbf{1}), \alpha]$. Define function $h : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ by $h(\alpha) := D_\infty(\varphi_{A, \alpha}^c(C))$ for $\alpha \in \mathbf{R}_+$, which is piecewise linear continuous. We show that h

is nonincreasing. It suffices to verify that the right derivative of h at $\alpha = 0$ is nonpositive. Let $i(C)$ be the set of indices j such that $\varphi_{A,\epsilon}^c(p_j) \neq p_j$ for every $\epsilon > 0$. For a small $\epsilon^* > 0$, let $C^* = (p_1^*, p_2^*, \dots, p_n^*) := \varphi_{A,\epsilon^*}^c(C)$. Then $p_j^* = p_j + \epsilon^*(\mathbf{1}_{A^c}, -\mathbf{1})$ if $j \in i(C)$, and $p_j^* = p_j$ otherwise. We show $D_\infty(C^*) \leq D_\infty(C)$. We may assume that there are indices k, l such that $k-1, l+1 \notin i(C)$ and $k, k+1, \dots, l-1, l \in i(C)$, where the indices are taken modulo n . For $j = k, k+1, \dots, l-1$, we have $D_\infty(p_j, p_{j+1}) = D_\infty(p_j^*, p_{j+1}^*)$. Consider $D_\infty(p_l^*, p_{l+1}^*) = D_\infty(p_l + \epsilon^*(\mathbf{1}_{A^c}, -\mathbf{1}), p_{l+1})$. Since $\varphi_A^c(p_{l+1}) = p_{l+1}$, there is an edge $s^c t^r$ in $K(p_{l+1})$ connecting $S^c \setminus A^c$ and S^r . Then $\epsilon^* \leq p_l(s^c) + p_l(t^r) - \mu(s, t) = p_l(s^c) + p_l(t^r) - p_{l+1}(s^c) - p_{l+1}(t^r) = p_l(t^r) - p_{l+1}(t^r) - p_{l+1}(s^c)$, where we use $p_{l+1}(s^c) + p_{l+1}(t^r) = \mu(s, t)$ by $s^c t^r \in EK(p_{l+1})$ and $p_l(s^c) = 0$ (since s^c is isolated in $K(p_l)$). By Lemma 2.2 and $p_l(t^r) - p_{l+1}(t^r) \geq \epsilon^*$, we have

$$D_\infty(p_l^*, p_{l+1}^*) = \max\{p_l(t^r) - \epsilon^* - p_{l+1}(t^r) \mid t^r \in S^r\} = D_\infty(p_l, p_{l+1}) - \epsilon^*.$$

Next consider $D_\infty(p_{k-1}^*, p_k^*) = D_\infty(p_{k-1}, p_k + \epsilon^*(\mathbf{1}_{A^c}, -\mathbf{1}))$. By Lemma 2.2 again, we have

$$D_\infty(p_{k-1}^*, p_k^*) = \max\{0, p_{k-1}(t^r) - (p_k(t^r) - \epsilon^*) \mid t^r \in S^r\} \leq D_\infty(p_{k-1}, p_k) + \epsilon^*.$$

Thus $\sum_{i=k-1}^l D_\infty(p_i, p_{i+1}) \geq \sum_{i=k-1}^l D_\infty(p_i^*, p_{i+1}^*)$. Consequently we obtain $D_\infty(C^*) \leq D_\infty(C)$.

Proof of Lemma 2.7. It suffices to show (1-2) for one particular balanced section R . Let $C = (p_1, p_2, \dots, p_n)$ be a cycle in Q_μ . Since any translation of R preserves D_∞ -distances among points in R , we may assume (*1) $(\varphi_R(p))^c > p^c$ for any point p in C . We use the same idea as above. For $\alpha \geq 0$, let $\varphi_{R,\alpha}$ be a map defined by

$$\varphi_{R,\alpha}(p) = p + (\mathbf{1}, -\mathbf{1}) \max\{\epsilon \in [0, \alpha] \mid p^c + \epsilon \mathbf{1} \leq (\varphi_R(p))^c\}.$$

Then $\varphi_R = \lim_{\alpha \rightarrow \infty} \varphi_{R,\alpha}$. Let $i(C)$ be the set of indices j with $\varphi_{R,\alpha}(p) \neq p$ for every $\alpha > 0$. For a small $\epsilon^* > 0$, let $C^* = (p_1^*, p_2^*, \dots, p_n^*) := \varphi_{R,\epsilon^*}(C)$. It suffices to show $D_\infty(C^*) \leq D_\infty(C)$. We may assume that there are indices k, l such that $k-1, l+1 \notin i(C)$ and $k, k+1, \dots, l-1, l \in i(C)$, where the indices are taken modulo n . For $j = k, k+1, \dots, l-1$, we have $D_\infty(p_j, p_{j+1}) = D_\infty(p_j^*, p_{j+1}^*)$. Then $D_\infty(p_l^*, p_{l+1}^*) = D_\infty(p_l + \epsilon^*(\mathbf{1}, -\mathbf{1}), p_{l+1}) = \max_{s^c \in S^c} \{p_{l+1}(s^c) - p_l(s^c) - \epsilon^*, 0\} = \max_{s^c \in S^c} \{p_{l+1}(s^c) - p_l(s^c), 0\} - \epsilon^*$. Here the last equality follows from $(p_{l+1})^c \not\leq (p_l^*)^c$ by the assumption (*1) and the balancedness of R (and Lemma 2.2 (2)). Moreover, we have (*2) $D_\infty(p_{k-1}^*, p_k^*) = D_\infty(p_{k-1}, p_k + \epsilon^*(\mathbf{1}, -\mathbf{1})) = \max\{p_k(s^c) + \epsilon^* - p_{k-1}(s^c), 0\} \leq \max\{p_k(s^c) - p_{k-1}(s^c), 0\} + \epsilon^*$. Hence we have $D_\infty(C^*) \leq D_\infty(C)$. Here the last inequality in (*2) holds strictly if and only if $p_{k-1}^c > p_k^c$, i.e., $\{p_{k-1}, p_k\}$ is not balanced. Consequently we have the claim (1-2).

2.3 Geodesics and embedding

A *path* P in $\mathbf{R}^{S^{cr}}$ is the image of a continuous map $\varrho : [0, 1] \rightarrow \mathbf{R}^{S^{cr}}$ (with respect to the Euclidean topology). Its *length from* $\varrho(0)$ *to* $\varrho(1)$ is defined by

$$\sup \left\{ \sum_{i=0}^{n-1} D_\infty(\varrho(t_i), \varrho(t_{i+1})) \right\},$$

where the supreme is taken over all $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n = 1$ and $n \geq 1$. Any segment $[p, q]$ has the length $D_\infty(p, q)$ from p to q since $D_\infty(p, q) = D_\infty(p, r) + D_\infty(r, q)$ for $r \in [p, q]$. A subset R in $\mathbf{R}^{S^{cr}}$ is said to be *geodesic* if for each pair $p, q \in R$ there exists a path in R of length $D_\infty(p, q)$ from p to q .

Proposition 2.9. T_μ , Q_μ , Q_μ^+ , and any balanced section $R \subseteq Q_\mu$ are all geodesic.

Proof. We show a general statement:

(*) For $R \subseteq R' \subseteq \mathbf{R}^{S^{cr}}$, suppose that R' is geodesic and there is a cyclically nonexpansive retraction φ from R' to R . Then R is also geodesic.

For $p, q \in R$, take a path P connecting p and q in R' of length $D_\infty(p, q)$ from p to q . Suppose that P is the image of $\varrho : [0, 1] \rightarrow R'$ with $(\varrho(0), \varrho(1)) = (p, q)$. The image $\varphi(P)$ of the cyclically nonexpansive retraction φ is also a path in R connecting p and q ; see Remark 2.8. We show that $\varphi(P)$ has length $D_\infty(p, q)$ from p to q . For $n > 0$ and $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n = 1$, consider cycle $C = (\varrho(t_0), \varrho(t_1), \dots, \varrho(t_n))$. Then $D_\infty(\varphi(C)) \leq D_\infty(C)$. Since $(p, q) = (\varrho(t_0), \varrho(t_n)) = (\varphi \circ \varrho(t_0), \varphi \circ \varrho(t_n))$, we have

$$D_\infty(p, q) \leq \sum_{i=0}^{n-1} D_\infty(\varphi \circ \varrho(t_i), \varphi \circ \varrho(t_{i+1})) \leq \sum_{i=0}^{n-1} D_\infty(\varrho(t_i), \varrho(t_{i+1})) = D_\infty(p, q).$$

Thus $\varphi(P)$ has the length $D_\infty(p, q)$ from p to q , and R is geodesic. Since P_μ is geodesic by convexity, Lemmas 2.5, 2.6, and 2.7 imply that T_μ , Q_μ^+ , and any balanced section in Q_μ^+ are geodesic.

Next we show that an arbitrary balanced section R in Q_μ is geodesic. Take any balanced section R' in Q_μ^+ . Then the restriction of $\varphi_{R'}$ to R is a bijection, and the inverse map is given by φ_R . For $p, q \in R$, take a path $P \subseteq R'$ connecting $\varphi_{R'}(p)$ and $\varphi_{R'}(q)$ of length $D_\infty(\varphi_{R'}(p), \varphi_{R'}(q))$ from $\varphi_{R'}(p)$ to $\varphi_{R'}(q)$. Consider the image $\varphi_R(P)$, which is a path connecting p and q . By Lemma 2.7, $D_\infty(q, p)$ plus the length of $\varphi_R(P)$ from p to q is equal to $D_\infty(\varphi_{R'}(q), \varphi_{R'}(p))$ plus the length $D_\infty(\varphi_{R'}(p), \varphi_{R'}(q))$ of P from $\varphi_{R'}(p)$ to $\varphi_{R'}(q)$. Since $D_\infty(q, p) + D_\infty(p, q) = D_\infty(\varphi_{R'}(q), \varphi_{R'}(p)) + D_\infty(\varphi_{R'}(p), \varphi_{R'}(q))$, the length of $\varphi_R(P)$ from p to q coincides with $D_\infty(p, q)$. Thus R is geodesic.

Finally, we show that Q_μ is geodesic. Take a pair p, q in Q_μ . If $\{p, q\}$ is balanced, then take a balanced section containing p, q and apply the argument above. So suppose that $\{p, q\}$ is not balanced. Then we can take a point $r \in Q_\mu$ such that $\{p, r\}$ is balanced, $\bar{r} = \bar{q}$, and $D_\infty(p, r) + D_\infty(r, q) = D_\infty(p, q)$. Take a path $P \subseteq Q_\mu$ connecting p, r of length $D_\infty(p, r)$ from p to r . Consider path $P \cup [r, q] \subseteq Q_\mu$, which connects p and q with length $D_\infty(p, q)$ from p to q . \square

Next we show that μ is realized by D_∞ -distances among a certain subsets in T_μ , Q_μ^+ , and any balanced section R in Q_μ^+ . Further we denote by $T_{\mu, s}$, $Q_{\mu, s}^+$, and R_s , the set of points p with $p(s^c) = p(s^r) = 0$ in T_μ , Q_μ^+ , and R , respectively. For each $s \in S$, we define three vectors $\mu_s, \mu_s^{in}, \mu_s^{out} \in \mathbf{R}^{S^{cr}}$ by

$$\begin{aligned} (\mu_s(t^c), \mu_s(t^r)) &= (\mu(t, s), \mu(s, t)), \\ (\mu_s^{in}(t^c), \mu_s^{in}(t^r)) &= (\mu(t, s), \max_{u \in S} \{\mu(u, t) - \mu(u, s)\}), \\ (\mu_s^{out}(t^c), \mu_s^{out}(t^r)) &= (\max_{u \in S} \{\mu(t, u) - \mu(s, u)\}, \mu(s, t)) \quad (t \in S). \end{aligned}$$

Both μ_s^{in} and μ_s^{out} belong to $T_{\mu, s}$. In particular μ_s^{in} and μ_s^{out} are unique maximal points in $T_{\mu, s}$ with respect the coordinates in S^c and in S^r , respectively. Now the embedding properties of T_μ and Q_μ^+ are summarized as follows:

Proposition 2.10. *The following properties hold:*

(1) $T_{\mu, s} = Q_{\mu, s}^+ = R_s$ for each $s \in S$ and any balanced section $R \subseteq Q_\mu^+$.

(2) $(p(s^c), p(s^r)) = (D_\infty(T_{\mu,s}, p), D_\infty(p, T_{\mu,s})) = (D_\infty(\mu_s^{out}, p), D_\infty(p, \mu_s^{in}))$ for $p \in T_{\mu}$ and $s \in S$.

(3) $\mu(s, t) = D_\infty(T_{\mu,s}, T_{\mu,t}) = D_\infty(\mu_s^{out}, \mu_t^{in})$ for all $s, t \in S$.

(4) If μ is a directed metric, then $T_{\mu,s} = \{\mu_s\}$ for each $s \in S$.

The property (2) means that the coordinate $p(s^c)$ (resp. $p(s^r)$) is the D_∞ -distance from $T_{\mu,s}$ to p (resp. p to $T_{\mu,s}$). The point μ_s^{in} plays a role of an *entrance* of $T_{\mu,s}$; every point can enter $T_{\mu,s}$ through μ_s^{in} . The point μ_s^{out} is an *exit* of $T_{\mu,s}$. The property (3) means that distance μ can be realized by D_∞ -distances among $T_{\mu,s}$. By (4), if μ is a metric, then each $T_{\mu,s}$ is a single point, and (S, μ) is isometrically embedded into (T_μ, D_∞) , (Q_μ^+, D_∞) , and (R, D_∞) by the map $s \mapsto \mu_s$.

Proof. (1). Let p be a point in $T_{\mu,s}$. By $p(s^c) = p(s^r) = 0$, if $p(u) = 0$, then u is incident to s^c or s^r in $K(p)$. Thus $K(p)$ has no isolated vertex, which implies $p \in Q_\mu^+$. Moreover, for any $\epsilon > 0$, the point $p \pm \epsilon(\mathbf{1}, -\mathbf{1})$ has a negative coordinate in s^c or s^r . Thus $p \pm \epsilon(\mathbf{1}, -\mathbf{1}) \notin Q_\mu^+$. This means that every section in Q_μ^+ contains $T_{\mu,s}$.

(2). Since every $q \in T_{\mu,s}$ satisfies $q(s^c) = 0$, we have $p(s^c) = p(s^c) - q(s^c) \leq D_\infty(q, p)$. Thus, it follows that $p(s^c) \leq D_\infty(T_{\mu,s}, p) \leq D_\infty(\mu_s^{out}, p)$. Conversely we have $\mu_s^{out}(t^r) - p(t^r) = \mu(s, t) - p(t^r) \leq p(s^c)$ for all $t^r \in S^r$, which implies $D_\infty(T_{\mu,s}, p) \leq \|((\mu_s^{out})^r - p^r)_+\|_\infty \leq p(s^c)$ (by Lemma 2.2 (1)).

(3). Since $p(s^c) = 0$ for all $p \in T_{\mu,s}$ and $q(t^c) \geq \mu(s, t)$ for all $q \in T_{\mu,s}$, we have $D_\infty(T_{\mu,s}, T_{\mu,t}) \geq \mu(s, t)$. Conversely, by (2), $D_\infty(T_{\mu,s}, T_{\mu,t}) \leq D_\infty(\mu_s^{out}, \mu_t^{in}) = \mu_s^{in}(t^c) = \mu(s, t)$.

(4). Take $p \in T_{\mu,s}$. For each $t \in S$, we have $p(t^c) \geq \mu(t, s)$ and $p(t^r) \geq \mu(s, t)$. Since $p(t^c) + p(u^r) = \mu(t, u)$ holds for some u , we have $p(t^c) = \mu(t, u) - p(u^r) \leq \mu(t, u) - \mu(s, u) \leq \mu(t, s)$, where we use the triangle inequality for the last inequality. Thus $p^c = (\mu_s)^c$, and similarly $p^r = (\mu_s)^r$. \square

Remark 2.11. $T_{\mu,s}$ is also contractible since it is a retract of $\{p \in P_\mu \mid p(s^c) = p(s^r) = 0\}$ by ϕ in Lemma 2.5.

2.4 Tight extensions, cyclically tight extensions, and congruence

The goal of this section is to show that the tight span and (a fiber of) the tropical polytope have universal embedding properties for metric extensions.

Let (S, μ) be a directed metric space. A directed metric space (V, d) is called an *extension* of (S, μ) if $S \subseteq V$ and $d(s, t) = \mu(s, t)$ for $s, t \in S$. An extension (V, d) is said to be *tight* if there is no extension (V, d') with $d' \leq d$ and $d' \neq d$. An extension (V, d) is said to be *cyclically tight* if there is no extension (V, d') of (S, μ) such that $d'(C) \leq d(C)$ for all cycles C in V and $d'(C') < d(C')$ for some cycle C' . For convention, we also use the notions of extension and tightness for metric functions, such as “ d is a cyclically tight extension of μ .”

By Proposition 2.10 (3-4), (T_μ, D_∞) , (Q_μ^+, D_∞) , and (R, D_∞) for any balanced section $R \subseteq Q_\mu^+$ are all extensions of (S, μ) . For an extension (V, d) of (S, μ) and $x \in V$, we define vector $d_x \in \mathbf{R}^{S^{cr}}$ by

$$(d_x(s^c), d_x(s^r)) = (d(s, x), d(x, s)) \quad (s \in S).$$

The main result of this section is the following:

Theorem 2.12. *Let (S, μ) be a directed metric space, and (V, d) its extension.*

- (1) d is tight if and only if there exists an isometric embedding $\rho : V \rightarrow T_\mu$ such that $\rho(s) = \mu_s$ for $s \in S$.
- (2) d is cyclically tight if and only if there exists an isometric embedding $\rho : V \rightarrow Q_\mu^+$ such that $\rho(s) = \mu_s$ for $s \in S$ and $\rho(V)$ is balanced.

Moreover, if tight, then such an isometric embedding ρ coincides with $x \mapsto d_x$ ($x \in V$).

In particular, (1) says that (T_μ, D_∞) is a unique inclusion-maximal tight extension of (S, μ) . (2) says that Q_μ^+ plays a role of the tight span with respect to the cyclic tightness. In contrast to tight extensions, there are many maximal cyclically tight extensions, which correspond to balanced sections in Q_μ^+ .

Proof. (1). Suppose that d is tight. Define $\rho : V \rightarrow \mathbf{R}^{S^c}$ by $\rho(x) = d_x$ ($x \in V$). Then $\rho(s) = \mu_s \in T_\mu$ for $s \in S$. By triangle inequality, we have $D_\infty(\rho(x), \rho(y)) = \max\{\|(d_y^c - d_x^c)_+\|_\infty, \|(d_x^r - d_y^r)_+\|_\infty\} \leq d(x, y)$. By tightness, we have $D_\infty(\rho(x), \rho(y)) = d(x, y)$. Since $d_x(s) + d_x(t) = d(s, x) + d(x, t) \geq d(s, t) = \mu(s, t)$, each $\rho(x) = d_x$ belongs to P_μ . Furthermore each d_x belongs to T_μ . Consider a nonexpansive retraction ϕ in Lemma 2.5. By Proposition 2.10 (2), $(D_\infty(\mu_s, \phi(d_x)), D_\infty(\phi(d_x), \mu_s)) = (\phi(d_x)(s^c), \phi(d_x)(s^r)) \leq (d(s, x), d(x, s))$. By tightness, $\phi(d_x) = d_x$ and thus $d_x \in T_\mu$ for $x \in V$.

We next show the if part. Let $\rho : V \rightarrow T_\mu$ be an isometric embedding. By Proposition 2.10 (2), we have $(\rho(x)(s^c), \rho(x)(s^r)) = (D_\infty(\mu_s, \rho(x)), D_\infty(\rho(x), \mu_s)) = (d(s, x), d(x, s)) = (d_x(s^c), d_x(s^r))$ for $s \in S$. Thus $\rho(x) = d_x$ for $x \in V$. Take a tight extension d' on V with $d' \leq d$. Each d'_x belongs to P_μ as above. Since $d'_x \leq d_x = \rho(x)$ and $\rho(x)$ is minimal in P_μ , we have $d'_x = \rho(x)$. Consequently $d'(x, y) = D_\infty(d'_x, d'_y) = D_\infty(\rho(x), \rho(y)) = d(x, y)$, and therefore d is tight.

(2). Suppose that d is cyclically tight. Since d is tight, by (1) it suffices to show that $\{d_x \mid x \in V\}$ is balanced and is a subset in Q_μ^+ . By Lemma 2.7, $\{d_x \mid x \in V\}$ is necessarily balanced. We verify $d_x \in Q_\mu^+$ for each $x \in V$. Suppose that $K(d_x)$ has an isolated vertex $s^c \in S^c$ (or $s^r \in S^r$). By the construction of map φ in Lemma 2.6, $D_\infty(d_s, d_x) + D_\infty(d_x, d_s) (= d_x(s^c) + d_x(s^r))$ strictly decreases. A contradiction to the cyclic tightness of d .

We next show the if part. We show that d is cyclically tight. Take a cyclically tight extension d' on V such that $d'(C) \leq d(C)$ for every cycle in V . By Lemma 2.7, it suffices to show $\bar{d}_x = \bar{d}'_x$ for $x \in V$. Since $d'(s, x) + d'(x, t) + d'(t, s) \leq d(s, x) + d(x, t) + d(t, s)$ for all $s, t \in S$ and $x \in V$, we have $d'_x(s^c) + d'_x(t^r) \leq d_x(s^c) + d_x(t^r)$, that is $d'_x(s^c) - d_x(s^c) \leq d_x(t^r) - d'_x(t^r)$. In particular, for an edge $s^c t^r$ in $K(d_x)$, we have $\mu(s, t) \leq d'_x(s^c) + d'_x(t^r) \leq d_x(s^c) + d_x(t^r) = \mu(s, t)$, and hence $d'_x(s^c) - d_x(s^c) = d_x(t^r) - d'_x(t^r)$. Therefore, for two edges $s^c t^r$ and $\tilde{s}^c \tilde{t}^r$ in $K(d_x)$, we have (*) $d'_x(s^c) - d_x(s^c) \leq d_x(\tilde{t}^r) - d'_x(\tilde{t}^r) = d'_x(\tilde{s}^c) - d_x(\tilde{s}^c) \leq d_x(t^r) - d'_x(t^r) = d'_x(s^c) - d_x(s^c)$. Since $K(d_x)$ has no isolated vertex, it follows from (*) that $d_x - d'_x \in (\mathbf{1}, -\mathbf{1})\mathbf{R}$. \square

Two directed metrics d, d' on the same set V are said to be *congruent* if $d(C) = d'(C)$ for each cycle C in V . Then d and d' are congruent if and only if there is $\alpha : V \rightarrow \mathbf{R}$ such that

$$d(x, y) = d'(x, y) - \alpha(x) + \alpha(y) \quad (x, y \in V).$$

This is a simple consequence of the well-known fact in the network flow theory that any circulation in a directed network is the sum of the incidence vectors of cycles; see [1].

The tropical polytope \bar{Q}_μ describes all congruence classes of cyclically tight extensions. This establishes a role of tropical polytopes in metric extensions. For map $\rho : V \rightarrow Q_\mu$, let $\bar{\rho} : V \rightarrow \bar{Q}_\mu$ be the map obtained by projecting $\rho(\cdot)$ to \bar{Q}_μ .

Proposition 2.13. *Let (S, μ) be a directed metric space, and let $V \supseteq S$.*

- (1) *Two cyclically tight extensions d, d' of μ on V are congruent if and only if $\bar{d}_x = \bar{d}'_x$ for each $x \in V$.*
- (2) *A directed metric d on V is congruent to a cyclically tight extension of μ if and only if there exists an isometric embedding $\rho : V \rightarrow Q_\mu$ such that $\bar{\rho}(s) = \bar{\mu}_s$ for $s \in S$ and $\rho(V)$ is balanced.*

Proof. (1). The if part immediately follows from the previous theorem and Lemma 2.7; take a balanced section R containing $\{d_x \mid x \in V\}$, and apply φ_R to $\{d'_x \mid x \in V\}$. Observe that $\bar{d}_x = \bar{d}'_x$ if and only if $d(s, x) - d'(s, x) = d'(x, t) - d(x, t)$ for all $s, t \in S$. Therefore, $\bar{d}_x \neq \bar{d}'_x$ implies $d(s, x) + d(x, t) + d(t, s) \neq d'(s, x) + d'(x, t) + d'(t, s)$ for some $s, t \in S$. This proves the only-if part.

(2). We first show the if part. Take any balanced section R in Q_μ^+ (with help of Lemma 2.4). Consider $\varphi_R \circ \rho$. The corresponding metric d' on V defined as $d'(x, y) = D_\infty(\varphi_R \circ \rho(x), \varphi_R \circ \rho(y))$ is a cyclically tight extension of μ . By Lemma 2.7, d and d' are congruent. Next we show the only-if part. Suppose that d is congruent to a cyclically tight extension d' of μ . Then there is $\alpha \in \mathbf{R}^V$ such that $d(x, y) - d'(x, y) = -\alpha(x) + \alpha(y)$ for $x, y \in V$. Let $\rho : V \rightarrow \mathbf{R}^{S^{cr}}$ be a map defined by

$$\rho(x) = d'_x + \alpha(x)(\mathbf{1}, -\mathbf{1}) \quad (x \in V).$$

Let d'' be the metric on V defined as $d''(x, y) = D_\infty(\rho(x), \rho(y))$. We claim $d'' \leq d$. Indeed, we have $D_\infty(\rho(x), \rho(y)) = \max_{s \in S} (d'(s, y) + \alpha(y) - d'(s, x) - \alpha(x))_+ \leq d'(x, y) + \alpha(y) - \alpha(x) = d(x, y)$ (by triangle inequality). If $\rho(V)$ is not balanced, then there is a cycle C in V such that $d'(C) < d''(C) \leq d(C)$. This is a contradiction to $d'(C) = d(C)$. Thus $\rho(V)$ is balanced and $d = d''$ holds. \square

2.5 Dimension criteria

Here we present matching-type criteria for dimensions of tight span T_μ and tropical polytope \bar{Q}_μ . For tropical polytopes, such a criterion has already been given by [4].

For two n -element subsets $A^c \subseteq S^c$ and $B^r \subseteq S^r$, let K_{A^c, B^r} be the complete bipartite graph with the bipartition (A^c, B^r) . A *matching* is a subset of edges which have no common vertices. A matching M is said to *perfect* if all vertices are covered by M . For a matching M , let $\mu(M) = \sum_{a^c b^r \in M} \mu(a, b)$. We consider the following maximum matching problems:

MT: Maximize $\mu(M)$ over all matchings M in K_{A^c, B^r} .

PMT: Maximize $\mu(M)$ over all perfect matchings M in K_{A^c, B^r} .

Since μ is nonnegative, the maximum values of two problems are same. Our interest is how the maximum is attained.

Theorem 2.14. *For a positive integer n , the following two conditions are equivalent:*

- (a) $\dim T_\mu \geq n$.
- (b) *There exist two n -element subsets $A^c \subseteq S^c$ and $B^r \subseteq S^r$ such that the maximum of MT is uniquely attained by a perfect matching M .*

Theorem 2.15 ([4, Theorem 4.2]). *For a positive integer n , the following two conditions are equivalent:*

- (a) $\dim \bar{Q}_\mu \geq n - 1$.
- (b) *There exist two n -element subsets $A^c \subseteq S^c$ and $B^r \subseteq S^r$ such that the maximum of PMT is uniquely attained.*

In [4, 5], the maximum integer n with property (b) is called the *tropical rank* of (the distance matrix of) $-\mu$. We give a proof sketch of Theorem 2.14; this proof method has already been established in [4] and [7, Appendix].

Sketch of the proof of Theorem 2.14. (a) \Rightarrow (b). Since $\dim T_\mu \geq n$, there exists $p \in T_\mu$ such that $K(p)$ has n components having no vertex s with $p(s) = 0$ (Lemma 2.3). Let M be a set of n edges obtained from each of the n components, and let $A^c \subseteq S^c$ and $B^r \subseteq S^r$ be the sets of vertices of M . By construction, M is a perfect matching of K_{A^c, B^r} . Then $\mu(M) = \sum_{a \in A^c} p(a) + \sum_{b \in B^r} p(b)$. Take any matching M' in K_{A^c, B^r} . Then $\mu(M') \leq \sum_{a \in A^c} p(a) + \sum_{b \in B^r} p(b)$ by $p(a) + p(b) \geq \mu(a, b)$. So the equality is attained exactly when $M = M'$.

(b) \Rightarrow (a). With help of the bipartite matching theory [14, Part II] and the strict complementary slackness theorem in linear programming, there is a positive vector $p^* : A^c \cup B^r \rightarrow \mathbf{R}_+$ such that $p^*(a^c) + p^*(b^r) = \mu(a, b)$ for $a^c b^r \in M$ and $p^*(a^c) + p^*(b^r) > \mu(a, b)$ otherwise. Therefore, by the same argument in Lemma 2.3, the set $T_{\mu; A^c, B^r}$ of minimal elements of polyhedron $\{p \in \mathbf{R}_+^{A^c \cup B^r} \mid p(a^c) + p(b^r) \geq \mu(a, b) \text{ } (a, b) \in A^c \times B^r\}$ has dimension at least n . Any $p \in T_{\mu; A^c, B^r}$ can be extended to $p : S^{cr} \rightarrow \mathbf{R}_+$ so that $p \in P_\mu$. Then decrease $p(u)$ for $u \notin A^c \cup B^r$ so that $p \in T_\mu$. Therefore the projection of T_μ to $\mathbf{R}^{A^c \cup B^r}$ includes $T_{\mu; A^c, B^r}$. This implies $\dim T_\mu \geq \dim T_{\mu; A^c, B^r} \geq n$. \square

3 Combinatorial characterizations of distances with dimension one

A *tree metric* is a metric that can be represented as the graph metric of some tree. The *tree metric theorem* [2, 16, 18] says that an undirected metric μ on a set S is a tree metric if and only if it satisfies the *four point condition*:

$$\mu(s, t) + \mu(u, v) \leq \max\{\mu(s, u) + \mu(t, v), \mu(s, v) + \mu(t, u)\} \quad (s, t, u, v \in S).$$

See also [15, Chapter 7]. Dress [6] interpreted the four point condition as a criterion for one-dimensionality of the undirected tight span of μ , and derived the tree metric theorem from the embedding property; see also [7].

In this section, we apply this idea to our directed versions of tight spans, and derive combinatorial characterizations of classes of distances realized by oriented trees. An *oriented tree* is a directed graph whose underlying undirected graph is a tree. For an oriented tree Γ and a nonnegative edge-length $\alpha : E\Gamma \rightarrow \mathbf{R}_+$, we define a directed metric $D_{\Gamma, \alpha}$ on $V\Gamma$ as follows. For two vertices $x, y \in V\Gamma$, let $P[x, y]$ denote the set of edges forming a unique path connecting x, y in the underlying undirected tree, and let $\vec{P}[x, y]$ be the set of edges in $P[x, y]$ whose direction is the same as the direction from x to y ; in particular $P[x, y] = \vec{P}[x, y] \cup \vec{P}[y, x]$ (disjoint union). Then let $D_{\Gamma, \alpha}(x, y) = \sum\{\alpha(e) \mid e \in \vec{P}[x, y]\}$. For a distance μ on S , an *oriented-tree realization* $(\Gamma, \alpha; \{F_s\}_{s \in S})$ consists of an oriented tree Γ , a positive edge-length $\alpha : E\Gamma \rightarrow \mathbf{R}_+$, and a family $\{F_s \mid s \in S\}$ of subtrees in Γ such that

$$\mu(s, t) = D_{\Gamma, \alpha}(F_s, F_t) \quad (s, t \in S),$$

where a *subtree* is a subgraph whose underlying undirected graph is connected, and $D_{\Gamma, \alpha}(F_s, F_t)$ denotes the shortest distance from F_s to F_t . A *directed path* is an oriented tree each of whose vertices has at most one leaving edge and at most one entering edge. The main results in this section are the following:

Theorem 3.1. *For a directed distance μ on S , the following conditions are equivalent:*

- (a) μ has an oriented-tree realization $(\Gamma, \alpha; \{F_s\}_{s \in S})$ so that Γ is a directed path.
- (b) $\dim T_\mu \leq 1$.
- (c) For $s, t, u, v \in S$ (not necessarily distinct), we have

$$\mu(s, u) + \mu(t, v) \leq \max\{\mu(s, v) + \mu(t, u), \mu(s, u), \mu(s, v), \mu(t, u), \mu(t, v)\}.$$

Theorem 3.2. *For a directed distance μ on S , the following conditions are equivalent:*

- (a) μ has an oriented-tree realization $(\Gamma, \alpha; \{F_s\}_{s \in S})$ so that each subtree F_s is a directed path.
- (b) $\dim \bar{Q}_\mu \leq 1$.
- (c) For $x, y, z, u, v, w \in S$ (not necessarily distinct), we have

$$\begin{aligned} & \mu(x, u) + \mu(y, v) + \mu(z, w) \\ & \leq \max \left\{ \begin{array}{ll} \mu(x, v) + \mu(y, u) + \mu(z, w), & \mu(x, v) + \mu(y, w) + \mu(z, u), \\ \mu(x, w) + \mu(y, u) + \mu(z, v), & \mu(x, w) + \mu(y, v) + \mu(z, u), \\ \mu(x, u) + \mu(y, w) + \mu(z, v) \end{array} \right\}, \end{aligned}$$

i.e., the tropical rank of $-\mu$ is at most 2.

Proof of Theorems 3.1 and 3.2. In both theorems, (b) \Leftrightarrow (c) follows from dimension criteria (Theorems 2.14 and 2.15). We prove (b) \Rightarrow (a) from embedding property (Proposition 2.10). Suppose first $\dim T_\mu \leq 1$. Then T_μ is a tree since T_μ is contractible (Remark 2.8). Recall (2.1). For each face (segment) $F = [p, q]$ in T_μ , there are $A, B \subseteq S$ such that either $q - p = D_\infty(p, q)(\mathbf{1}_{A^c}, -\mathbf{1}_{B^r})$ or $p - q = D_\infty(q, p)(\mathbf{1}_{A^c}, -\mathbf{1}_{B^r})$. Therefore subspace $([p, q], D_\infty)$ is isometric to the segment in (\mathbf{R}, D_∞^+) . Then an oriented-tree realization $(\Gamma, \alpha; \{F_s\}_{s \in S})$ is obtained by the following way. The underlying undirected tree of Γ is the 1-skeleton graph of T_μ . We orient each undirected edge pq as $p \rightarrow q$ when $D_\infty(p, q) > 0$ (and $D_\infty(q, p) = 0$). The edge-length $\alpha(pq)$ is given by $\max\{D_\infty(p, q), D_\infty(q, p)\}$. Consider $T_{\mu, s}$, which is also the union of segments and is contractible (Remark 2.11). Let F_s be its corresponding subgraph in Γ , which is a subtree. By Proposition 2.9, the restriction of (T_μ, D_∞) to the set of the vertices of T_μ is isometric to $(V\Gamma, D_{\Gamma, \alpha})$. By Proposition 2.10 (3), μ is realized by $(\Gamma, \alpha; \{F_s\}_{s \in S})$. Next we verify that Γ is a directed path. Suppose to the contrary that there is a vertex having two entering edges or two leaving edges. Then there exists $s, t \in S$ such that the unique path joining F_s and F_t passes through these two edges. Then both $\mu(s, t)$ and $\mu(t, s)$ are positive. This contradicts (c) for the case $(v, u) = (s, t)$.

Next suppose $\dim \bar{Q}_\mu \leq 1$. Consider the following section $R \subseteq Q_\mu^+$:

$$R = \{p \in Q_\mu^+ \mid \forall \epsilon > 0, p + \epsilon(\mathbf{1}, -\mathbf{1}) \notin Q_\mu^+\}.$$

Then R is balanced since it consists of points p such that p^r has a zero component. Also R is a subcomplex (the union of faces) of T_μ . Therefore R is also a tree each of whose

segments is isometric to a segment in (\mathbf{R}, D_∞^+) . Similarly, from the 1-skeleton of R we obtain an oriented-tree realization $(\Gamma, \alpha; \{F_s\}_{s \in S})$ of μ . We verify that each subtree F_s is a directed path. Suppose to the contrary that there is a vertex v in F_s such that v has two entering edges or two leaving edges in F_s . Then we can take a point p in $T_{\mu,s}$ such that $p(s^r) = D_\infty(p, T_{\mu,s}) = D_\infty(p, \mu_s^{in}) > 0$ or $p(s^c) = D_\infty(T_{\mu,s}, p) = D_\infty(\mu_s^{out}, p) > 0$, which contradicts $(p(s^c), p(s^r)) = (0, 0)$ (Proposition 2.10 (2)).

Next we show (a) \Rightarrow (b) or (c). Suppose that μ is realized by $(\Gamma, \alpha; \{F_s\}_{s \in S})$. It suffices to consider the case where each F_s is a single vertex. Indeed, suppose that F_s is a directed path of tail v^- and head v^+ . Let $S' = S \setminus s \cup \{s^-, s^+\}$, and let F_{s^-} and F_{s^+} be subtrees consisting of singletons v^- and v^+ , respectively. Consider the new distance μ' on S' by $\mu'(t, u) = D_{\Gamma, \alpha}(F_t, F_u)$ for $t, u \in S'$. Then the distance matrix of μ is obtained by deleting the s^- -th row and the s^+ -th column from μ' . Namely μ is a submatrix of μ' . By dimension criteria, $\dim T_\mu \leq \dim T_{\mu'}$ and $\dim \bar{Q}_\mu \leq \dim \bar{Q}_{\mu'}$. So suppose that each F_s is a single vertex; in particular μ is a metric. We first show (a) \Rightarrow (c) in the first theorem. Now μ can be regarded as D_∞^+ -distances among points x_s in the real line \mathbf{R} . Namely $\mu(s, t) = (x_t - x_s)_+$ for $s, t \in S$. Take any $s, t, u, v \in S$. We verify the condition in (c). If $x_u \leq x_s$, then $\mu(s, u) = 0$ and $\mu(s, u) + \mu(t, v) = \mu(t, v)$. So we may assume that $x_u > x_s$ and $x_v > x_t$. It suffices to consider three cases (i) $x_s < x_u \leq x_t < x_v$, (ii) $x_s \leq x_t < x_u \leq x_v$, and (iii) $x_s \leq x_t < x_v \leq x_u$. (i) implies $\mu(s, u) + \mu(t, v) \leq \mu(s, v)$. Both (ii) and (iii) imply $\mu(s, u) + \mu(t, v) = \mu(s, v) + \mu(t, u)$.

Finally we show (a) \Rightarrow (b) in the second theorem; we use an alternative approach to avoid case disjunctions. We claim:

(*) μ is congruent to some tree metric d .

Suppose that (*) is true. Then there is $\alpha : S \rightarrow \mathbf{R}$ such that $\mu(s, t) = d(s, t) - \alpha(s) + \alpha(t)$ ($s, t \in S$). One can easily see that Q_d is a translation of Q_μ . It is known that the dimension of the tropical polytope spanned by a tree metric is at most 1 [5, Theorem 28]. Therefore $\dim \bar{Q}_\mu$ is also at most 1.

We show the claim (*). Now μ is realized by $(\Gamma, \alpha; \{F_s\}_{s \in S})$ with each subtree F_s being a single vertex v_s . For an edge e in Γ , let (A_e, B_e) be the ordered bipartition of S such that s belongs to A_e if and only if v_s belongs to the connected component of $\Gamma \setminus e$ containing the tail of e . Let $\vec{\delta}_{A_e, B_e}$ be a directed metric on S defined by

$$\vec{\delta}_{A_e, B_e}(s, t) = \begin{cases} 1 & \text{if } (s, t) \in A_e \times B_e, \\ 0 & \text{otherwise,} \end{cases} \quad (s, t \in S).$$

By construction, we have $\mu = \sum_{e \in E\Gamma} \alpha(e) \vec{\delta}_{A_e, B_e}$. Then one can easily see that $\vec{\delta}_{A_e, B_e}$ is congruent to $(\vec{\delta}_{A_e, B_e} + \vec{\delta}_{B_e, A_e})/2$. Here $\vec{\delta}_{A_e, B_e} + \vec{\delta}_{B_e, A_e}$ coincides with the *split metric* of *split* (bipartition) $\{A_e, B_e\}$. So μ is congruent to a nonnegative sum of split metrics for the pairwise *compatible* family of splits $\{\{A_e, B_e\} \mid e \in E\Gamma\}$, which is just a tree metric; see [15]. \square

A directed metric μ is called a *directed tree metric* if it has an oriented-tree realization $(\Gamma, \alpha; \{F_s\}_{s \in S})$ such that each subtree F_s is a single vertex. For a directed metric μ , let μ^t denote the metric obtained by transposing μ , i.e., $\mu^t(x, y) := \mu(y, x)$. The following corollary sharpens [5, Theorem 28], and includes a nonnegative version of [13, Theorem 5].

Corollary 3.3. *For a directed metric μ on S , the following conditions are equivalent:*

(a) μ is a directed tree metric.

- (b) μ is congruent to a tree metric.
- (c) $\mu + \mu^t$ satisfies the four point condition and $\mu(x, y) + \mu(y, z) + \mu(z, x) = \mu(z, y) + \mu(y, x) + \mu(x, z)$ holds for every triple $x, y, z \in S$.
- (d) the tropical rank of $-\mu$ is at most 2.

Proof. In the proof of (b) \Rightarrow (a) in the previous theorems, if μ is a metric, then we can take F_s as a single vertex since $T_{\mu,s}$ is a single point μ_s (Proposition 2.10 (4)). So we obtain (a) \Rightarrow (b) \Rightarrow (d) \Rightarrow (a) from the proof above. We verify (b) \Leftrightarrow (c). Decompose μ as $(\mu + \mu^t)/2 + (\mu - \mu^t)/2$. Then the second condition in (c) is rephrased as $(\mu - \mu^t)(C) = 0$ for every cycle C consisting of three elements. One can easily see that this is equivalent to $(\mu - \mu^t)(C) = 0$ for every cycle C , i.e., μ is congruent to $(\mu + \mu^t)/2$. From this fact and the tree metric theorem, we obtain (b) \Leftrightarrow (c). \square

4 Multicommodity flows

Originally our directed version of tight spans was motivated by the multicommodity flow theory in combinatorial optimization; see [1, Section 17] and [14, Part VII]. In this section, we briefly sketch how tight spans and tropical polytopes are applied to the study of multicommodity flows in directed networks; see [8, 11] for applications of undirected tight spans to multicommodity flows in undirected networks.

By a *network* we mean a quadruple (V, E, S, c) consisting of a directed graph (V, E) , a specified vertex subset $S \subseteq V$, and a nonnegative integer-valued edge-capacity $c : E \rightarrow \mathbf{Z}_+$. We call a vertex in S a *terminal*. A directed path P in (V, E) is called an *S-path* if P connects distinct terminals in S . A *multiflow* (*multicommodity flow*) is a pair (\mathcal{P}, λ) of a set \mathcal{P} of *S*-paths and a nonnegative flow-value function $\lambda : \mathcal{P} \rightarrow \mathbf{R}_+$ satisfying the capacity constraint:

$$\sum \{\lambda(P) \mid P \in \mathcal{P}, P \text{ contains } e\} \leq c(e) \quad (e \in E).$$

Let $\mu : S \times S \rightarrow \mathbf{R}_+$ be a nonnegative weight defined on the set of all ordered pairs on terminals. For a multiflow $f = (\mathcal{P}, \lambda)$, its *flow-value* $\text{val}(\mu, f)$ is defined by $\sum_{P \in \mathcal{P}} \mu(s_P, t_P) \lambda(P)$, where s_P and t_P denotes the starting vertex and the end vertex of P , respectively. Namely $\mu(s, t)$ represents a value of a unit (s, t) -flow. The μ -weighted maximum multiflow problem is formulated as follows.

MCF: Maximize $\text{val}(\mu, f)$ over all multiflows f in (V, E, S, c)

This is one of fundamental problems in operations research, and has a wide range of practical applications; see [1, Section 17.1].

We regard μ as a directed distance on S . For a simplicity, we assume that μ is a *metric*. As is well-known in the multiflow theory [12], an LP-dual to (MCF) is a linear optimization over metrics:

$$\begin{aligned}
 (4.1) \quad & \text{Minimize} && \sum_{xy \in E} c(xy) d(x, y) \\
 & \text{subject to} && d \text{ is a directed metric on } V, \\
 & && d(s, t) = \mu(s, t) \quad (s, t \in S).
 \end{aligned}$$

Here recall notions in Section 2.4. This is nothing but a linear optimization over all extensions of (S, μ) . Since c is nonnegative, the minimum is always attained by a tight

extension. By Theorem 2.12 (1), possible candidates d for optimum are isometrically embedded into (T_μ, D_∞) . So the problem reduces to be an optimization over isometric embeddings ρ . Thus we have the following min-max relation sharpening the LP-duality.

Theorem 4.1. *Let (V, E, S, c) be a network and let μ be a directed metric on S . The following min-max relation holds:*

$$(4.2) \quad \max\{ \text{val}(\mu, f) \mid f: \text{multiflow in } (V, E, S, c) \}$$

$$= \min \left\{ \sum_{xy \in E} c(xy) D_\infty(\rho(x), \rho(y)) \mid \rho: V \rightarrow T_\mu, \rho(s) = \mu_s \ (s \in S) \right\}.$$

We give two interpretations of this minimization problem in RHS of (4.2) below. The first is a *facility location* on T_μ . There are facilities at points μ_s ($s \in S$) in T_μ . We are going to locate new facilities $x \in V \setminus S$ at points $\rho(x)$ in T_μ . Here facilities communicate with each other, and have the *communication cost*, which is a monotone function of their distances $D_\infty(\rho(x), \rho(y))$. The objective is to find a location of the minimum communication cost. In the literature of the location theory [17], the undirected version of this problem is known as *multifacility location problem*. The second interpretation comes from electrical circuits. We associate each vertex x with a T_μ -valued *potential* $\rho(x)$. The objective is to minimize the *energy*, a function of *potential differences* $D_\infty(\rho(x), \rho(y))$ measured in (T_μ, D_∞) , under the *boundary condition* $\rho(s) = \mu_s$.

Suppose the case where network (V, E, S, c) is *Eulerian*, i.e., for each vertex x , the sum of capacities over all edges entering x is equal to the sum of capacities over all edges leaving x . In this case, capacity function c is decomposed into the characteristic vectors of (possibly repeated) cycles C_1, C_2, \dots, C_m in (V, E) . For any metric d on V , the objective value in (4.1) is given by

$$d(C_1) + d(C_2) + \dots + d(C_m).$$

Therefore, *the minimum of (4.1) is always attained by a cyclically tight extension*. Moreover two congruent metrics have the same objective value. By Theorem 2.12 (2) and Proposition 2.13 (2), any metric d congruent to a cyclically tight extension is isometrically embedded into (Q_μ, D_∞) . Thus we have the following min-max relation:

Theorem 4.2. *Let (V, E, S, c) be an Eulerian network and let μ be a directed metric on S . Then the following min-max relation holds:*

$$(4.3) \quad \max\{ \text{val}(\mu, f) \mid f: \text{multiflow in } (V, E, S, c) \}$$

$$= \min \left\{ \sum_{xy \in E} c(xy) D_\infty(\rho(x), \rho(y)) \mid \rho: V \rightarrow Q_\mu, \bar{\rho}(s) = \bar{\mu}_s \ (s \in S) \right\}.$$

Here we can take ρ so that $\rho(V)$ lies on any fixed balanced section $R \subseteq Q_\mu$. So the RHS in (4.3) is essentially a facility location problem on the tropical polytope \bar{Q}_μ .

In the single commodity maximum flow problem, a classic theorem by Ford-Fulkerson says that the maximum flow value is equal to the minimum value of cut capacity, and there always exists an *integral* maximum flow, i.e., a maximum flow whose flow-value function is integer-valued; see [1, 14]. Such a *combinatorial min-max theorem* and an *integrality theorem* are closely related to the geometry of tight spans and tropical polytopes. This issue, however, is beyond the scope of the paper. So we will discuss it in the next paper [9].

Acknowledgments

The first author is supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan. The second author is supported by Nanzan University Patch Research Subsidy I-A-2 for the 2008 academic year.

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